Box Full of Mathematics

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Abstract: *The purpose of this paper is to describe the mathematics that emanates from the construction of an origami box. The design of this box was inspired by the design of a traditional origami masu box. We first construct a simple origami box from a rectangular sheet and then discuss some of the mathematical questions that arise in the context of geometry and algebra. The activity can be used as a context for illustrating how algebra and geometry, like other branches of mathematics, are interrelated.*

Keywords. *paper folding, origami, art, geometry, algebra*

1 Introduction

1.1 Why Study Origami?

The Common Core State Standards put a tremendous amount of emphasis on conceptual understanding (CCSSI, 2010). Origami provides a powerful context for conceptual understanding of mathematical ideas. Among other things, origami gives our students ready-made manipulatives that can be used to visualize abstract mathematical ideas in a concrete manner (Haga, 2006; Hull, 2006). For instance, when one creates a box from a rectangular sheet of paper, the box becomes the object that can be manipulated and analyzed, and abstract concepts like length, width, height, volume, and surface area become something that one can "touch." When students have objects that they have created, students communicate better with one another and with their teacher. Moreover, paper folding in general is mathematics in action. When one is folding paper, she or he is playing with mathematical concepts like perpendicular bisection, angle bisection, properties of right isosceles triangle, just to name a few of the mathematical concepts that are inextricably connected to paper folding (Tubis & Mills, 2006). In fact, it is fairly difficult to separate paper folding from mathematics. Due to the link between origami and art, origami can additionally be used to inspire artistic-minded students to think mathematically. Lastly, origami creates a powerful context for the application of several of Gardner's nine intelligences (Gardner, 2006; Wares, 2013), including linguistic, logical-mathematical, bodily-kinesthetic, spatial, interpersonal, intrapersonal, and existential intelligence.

1.2 Objectives

In this paper we learn to fold an origami box, and discuss the mathematics embedded in the box. No experience in origami is needed to construct this box. However, it is important to make the creases sharp and accurate. Figure 1 illustrates the two types of creases that are formed when a piece of paper is folded.

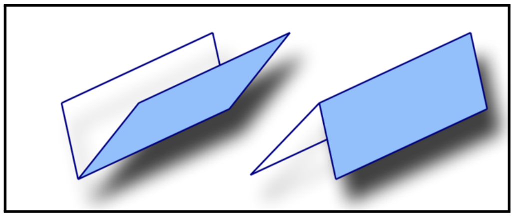

Fig. 1: *Valley crease and mountain crease.*

We need a rectangular sheet of paper to fold the box. An $8.5'' \times 11''$ rectangular sheet of paper straight out of the recycling bin will work. Figure 2 shows a photograph of the box that we will be making. The box is shaped like a prism with a square base. The box may resemble a cube, but it is not.

Fig. 2: *This is the box we will be making.*

Let us use the thirteen steps, labeled as steps (a) - (m) in Figures 3 and 4, to construct the box. In addition, a video illustrating how to fold the box is available at `https://youtu.be/vjCzf0hese0`.

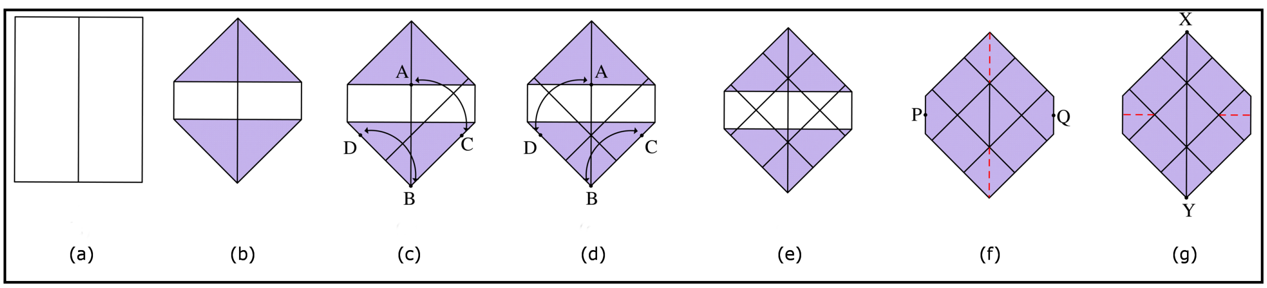

Fig. 3: *First seven steps (i.e., (a)-(g)) of the box construction process.*

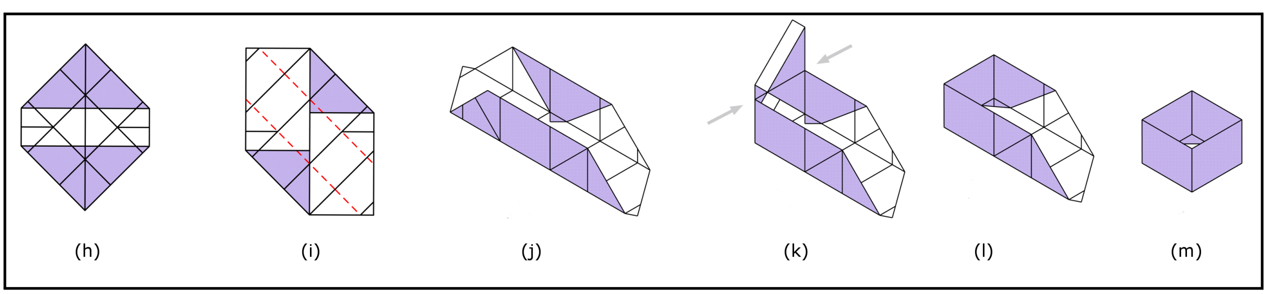

Fig. 4: *Last six steps (i.e., (h)-(m)) of the box construction process.*

2 Constructing the Box

Let us follow the thirteen steps to construct the box.

Step (a): Start with a sheet of paper that is 11 inches by 8.5 inches. With the white side up, make a valley crease that is parallel to the length of the rectangular sheet.

Step (b): Fold the left half and the right half of the top horizontal edge of the rectangular sheet over so that the two halves meet on the crease created in step (a). Fold the left half and the right half of the bottom horizontal edge of the rectangular sheet over so that the two halves meet on the crease created in Step (a). The folded paper should be hexagonal in shape.

Step (c): Drop the bottom-right slant edge of the hexagon onto point A, so that point B is dropped on the bottom-left edge of the hexagon. The new position of point B is labelled as point D, and point C is the point on the bottom-right slant edge of the hexagon that falls on point A. Unfold the paper back to its hexagonal shape. This step creates a slant crease parallel to the bottom-right edge of the hexagonal shape.

Step (d): Drop the bottom-left slant edge of the hexagon onto point A, so that point B is dropped on the bottom-right edge of the hexagon. The new position of point B is labelled as point C, and point D is the point on the bottom-left slant edge of the hexagon that falls on point A. Unfold the paper back to its hexagonal shape. This step creates a slant crease parallel to the bottom-left edge of the hexagonal shape.

Step (e): Rotate the hexagonal shape by an angle of $180°$ around the center of the hexagon and carry out Steps (c) and (d) one more time. This will allow you to create two more slant creases parallel to the edges.

Step (f): Flip the hexagonal shape over. By dropping point P on point Q, make two valley creases along the dotted line segments. Unfold the paper back to its hexagonal shape.

Step (g): By dropping point X on point Y, make two valley creases along the dotted line segments. Unfold the paper back to its hexagonal shape.

Step (h): Flip the shape over.

Step (i): Unfold the top left corner and the bottom right corner of the original rectangular sheet. By folding and then unfolding appropriately, make two valley creases along line segments marked with dotted lines.

Step (j): Lift the outer parts of the hexagon up along the creases created in Step (i) to form two of the walls of the box that will be constructed.

Step (k): By pushing the two short creases created in Steps (f) and (g) inward (as shown by the arrows), lift one of the flat ends of the structure up to form the next wall of the box that will be constructed.

Step (l): Tuck the excess paper inside.

Step (m): Repeat Steps (k) and (l) with the other flat end to complete the box.

3 Extending the Task

3.1 An Interesting Question

The design of the box that we have just constructed was inspired by the design of a traditional origami masu box. The box was designed by the author. A long time ago, wooden masu boxes were used to measure rice in Japan. A traditional origami masu box is made from a square sheet of paper, not a rectangular sheet. One of the obvious questions that comes to mind is how much rice would the constructed box hold, if we knew the dimensions of the original rectangular sheet? Out of curiosity, by looking at the crease marks, one may ask the following: *If the dimensions of the rectangular sheet are known, how can we determine the volume of the constructed box?* This is a large-scale geometry problem that can be solved using knowledge of high school geometry and algebra. Let us discuss a brief sketch of a solution. The author encourages the readers to come up with their own solutions before reading any further.

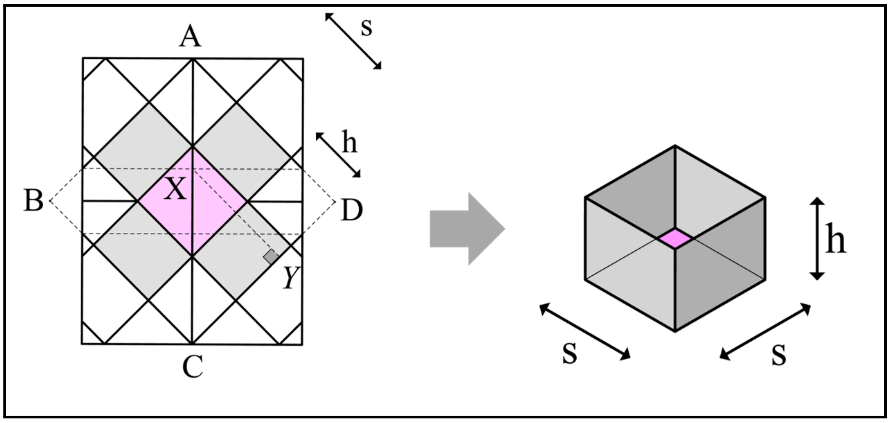

Fig. 5: *(left) The crease marks created by the folds; (right) Constructed box.*

3.2 An Algebraic Solution

Let the width and the length of the original rectangular sheet that was folded be a and b, respectively, with $a < b$, and $\frac{b}{a} < 1.5$ (when $\frac{b}{a} \geq 1.5$, the box does not fold properly). Let s and h be the length of the square base and the height of the constructed box, respectively (see Figure 5, right). Let us carefully transform the constructed box back into the rectangular sheet by carefully unfolding the box.

Figure 5 (left) shows the crease marks created by the folds. The dotted line segments are lines that are not a part of the crease marks in Figure 5 (right). The dotted lines will help us discuss the following solution. Line segment AC represents the crease created in Step (a). Moreover, $ABCD$ is a square with diagonal AC. Note points B and D lie outside the rectangular sheet. Point X is the point where the top two corners of the rectangular sheet meet on the crease represented by line segment AC. Line segment XY is drawn perpendicular to line segment CD. Due to the nature of the folds, the following conclusions may be drawn.

1. Since $\triangle ADC$ (in Figure 5 (left)) is a right isosceles triangle with hypotenuse, $AC = b$, $AD = \frac{b}{\sqrt{2}}$. Note the three angles of $\triangle ADC$ are $45°$, $45°$, and $90°$.

2. Since in Figure 5 (left), $AC = b$, $AX = \frac{a}{2}$, and $XC = AC - AX$, $XC = b - \frac{a}{2}$.

3. Since $\triangle XYC$ (in Figure 5 (left)) is a right isosceles triangle with hypotenuse $XC = b - \frac{a}{2}$, $XY = \frac{b}{\sqrt{2}} - \frac{a}{2\sqrt{2}}$.

4. Due to the nature of the folds, the height of the constructed box, h, must be half of the length of XY (see Figure 5). Hence the height, of the constructed box, h, is as follows: $h = \frac{XY}{2} = \frac{b}{2\sqrt{2}} - \frac{a}{4\sqrt{2}}$.

5. The length of the square base of the box, s, must be equal to $AD - 2h = AD - 2\left(\frac{b}{2\sqrt{2}} - \frac{a}{4\sqrt{2}}\right) = \frac{a}{2\sqrt{2}}$

6. Therefore, the volume of the constructed box is $h \times s^2 = \left(\frac{b}{2\sqrt{2}} - \frac{a}{4\sqrt{2}}\right)\left(\frac{a}{2\sqrt{2}}\right)^2 = \frac{a^2 b}{16\sqrt{2}} - \frac{a^3}{32\sqrt{2}}$.

3.2.1 Calculations for a standard $8.5'' \times 11''$ piece of paper

To find the volume of the box that was constructed, we can replace a with 8.5 and b with 11 in the above formula. The volume of the constructed box turns out to be as follows: $\frac{(8.5)^2(11)}{16\sqrt{2}} - \frac{(8.5)^3}{32\sqrt{2}} = \frac{(8.5)^2(13.5)}{32\sqrt{2}} \approx 21.55$ cubic inches.

4 In Conclusion

One of the strengths of paper folding activities is that they can be easy to understand and model, yet at the same time the solutions are not always obvious. Since students have a concrete object (the folded paper) in front of them, the level of intellectual engagement with the task is heightened and the quality of communication in the classroom becomes richer during the lesson. By its nature, origami creates a context for rich discourse.

Since mathematics is a cultural endeavor, it is ideal to teach mathematics as it manifests in the context of various cultures (D'Ambrosio, 2001). Appreciation of cultural diversity is not only important for the minority groups, but it is also important for the dominant ethnic group in any society because even the members of the dominant ethnic group will be working in an environment that is increasingly getting more and more diverse (Shirley, 2001). The use of origami in mathematics classrooms can provide a powerful context for the appreciation of cultural diversity of our world.

The Common Core (CCSSI, 2010) describes the following as the Standards for Mathematical Practice:

- Make sense of problems and persevere in solving them
- Reason abstractly and quantitatively
- Construct viable arguments and critique the reasoning of others
- Model with mathematics
- Use appropriate tools strategically
- Attend to precision
- Look for and make use of structure
- Look for and express regularity in repeated reasoning

The author believes all of the above Standards of Mathematical Practice can be implemented in a well-orchestrated and well-designed activity involving origami. Activity described in this paper incorporates most of the Standards for Mathematical Practice, if not all. More specifically, the origami activity described in this paper incorporates the following Standards for Mathematical Content in high school: seeing the structure in expressions (algebra), interpret functions that arise in applications in terms of the context (functions), build a function that models a relationship between two quantities (functions), explain volume formulas and use them to solve problems (geometry), visualize relationships between two-dimensional and three-dimensional objects (geometry), apply geometric concepts in modelling situations (geometry).

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